Hadron interactions, color and QCD partons

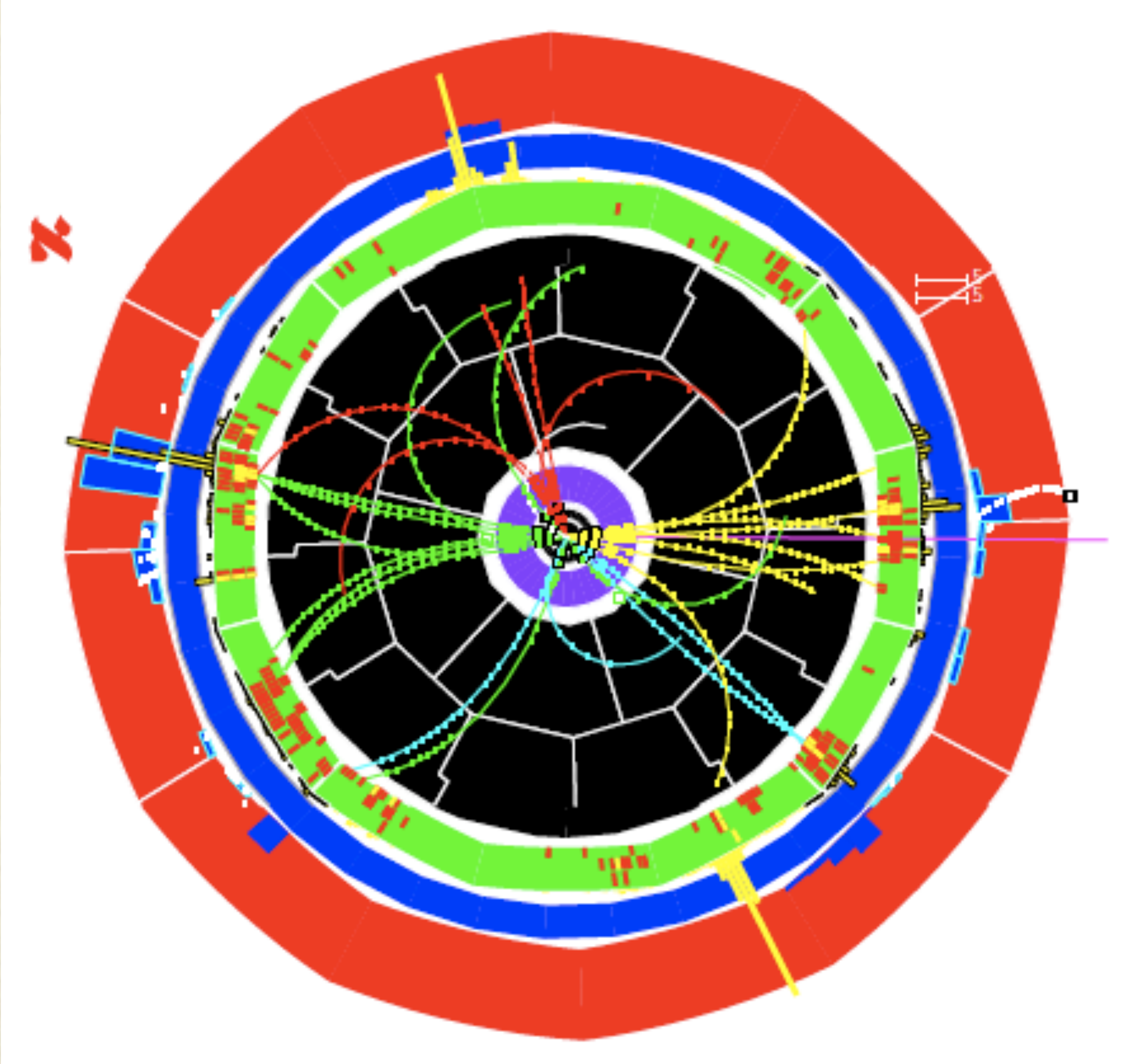
2. Jets, gluons and "gluers", LPHD and Quantum Mechanics (*coherence*)

Yuri Dokshitzer
LPTHE Jussieu, Paris
& PNPI, St Petersburg

Ecole Joliot-Curie
September-October 2013

The status of "***Jet Fragmentation***" as a subject can be looked upon as a successful symbiosis of ***astrophysics*** (**ideas**), ***astronomy*** (**observation**) and ***astrology*** (**inspiration**).

"***Successful***" - because it represents a rare example of a field in which we found ourselves capable of **predicting** ***so much*** while **understanding** ***so little***.



Aleph Higgs event $ZH \to q\bar{q}b\bar{b}$

One don't see no Higgses, no b-quarks

Just bunches of hadrons - "jets"...

To be able to read such messages one needs understanding of QCD

dark ages

Some forty years ago it was the epoch of ISR and SPEAR - the first Jet Labs.

In the realm of ***hard interactions***, Feynman invented his famous "***plateau***"- ***ln E*** hadrons streaming from a single quark-parton that is struck out of the target proton in a DIS process.

Moving from the high energy side (***soft interactions***) Gribov drew an energetic hadron fluctuating into ***ln E*** partons - the parton content of the ***Pomeron***.

The key word ***duality*** was pronounced in the context of the inter-connection between ***partons*** and ***hadrons***.

Those, however, were the times of pictures, of physical intuition, rather than theoretical expectations, let alone predictions. So that as late as 1976, three-jet pioneers (EGR) still spoke about logarithmic multiplicity as being merely "*fashionable*".

Approximately in the same time the HEP community started to believe, for earnest, in quarks as such ! The turn has occurred after the discovery of the 4-th - ***charmed*** - quark (***November revolution 1974***) predicted by Bjorken & Glashow back in 1964.

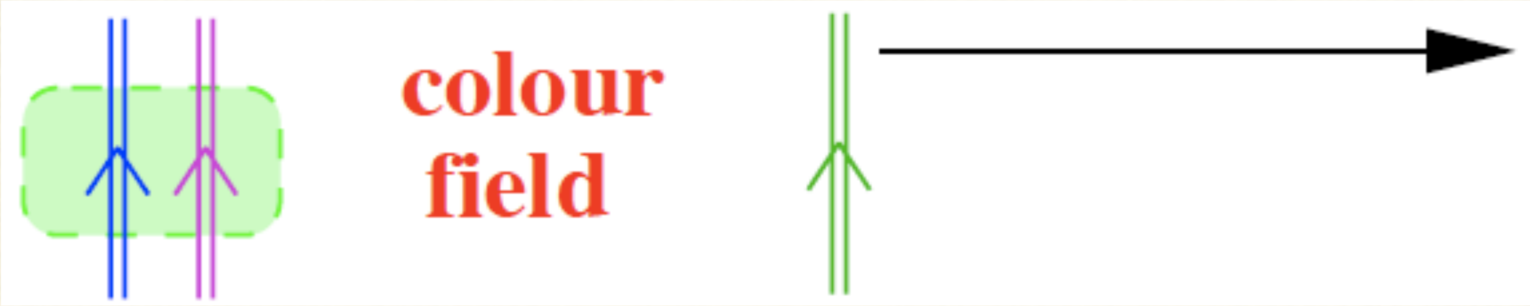
The notion of ***hadron jets*** goes back to the dark times of the dominance of cosmic rays...

On the theory side, the existence of jets was envisaged from "parton models" in the late 60's.

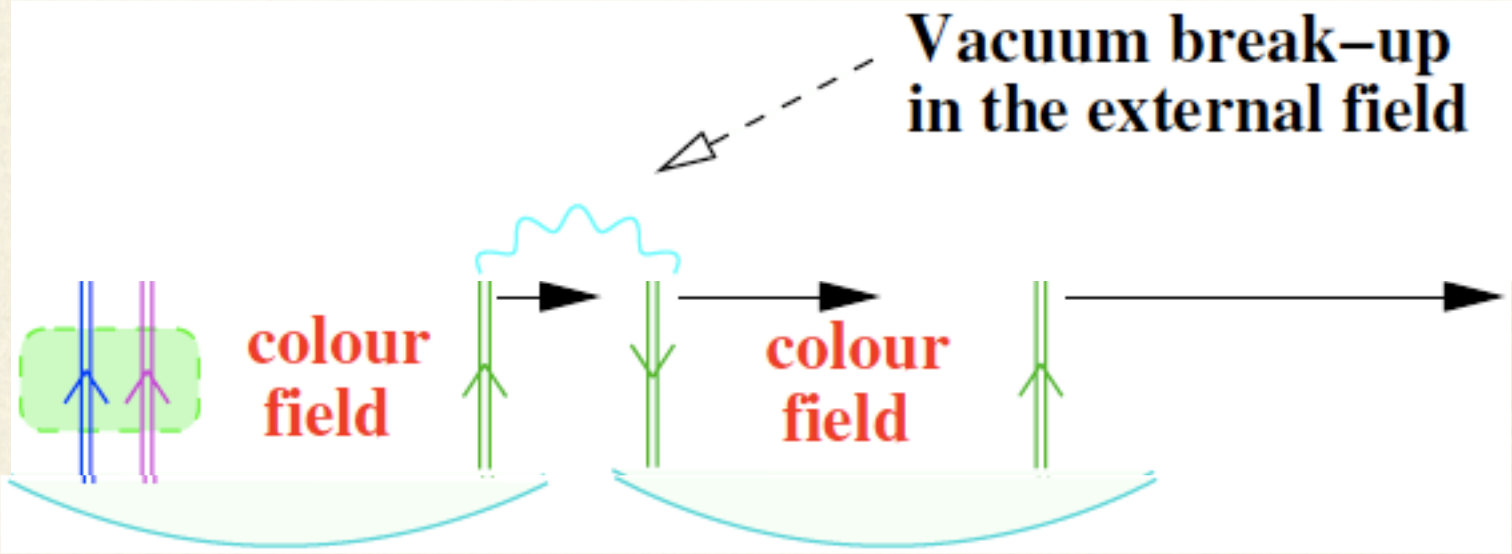
S.D.Drell, D.J.Levy & T.-M.Yan (1969-70) **N.Cabibbo, G.Parisi & M.Testa** (1970)

Kogut–Susskind vacuum breaking picture

- In a DIS a *green* quark in the proton is hit by a virtual photon
- The quark leaves the stage and the colour field starts to build up

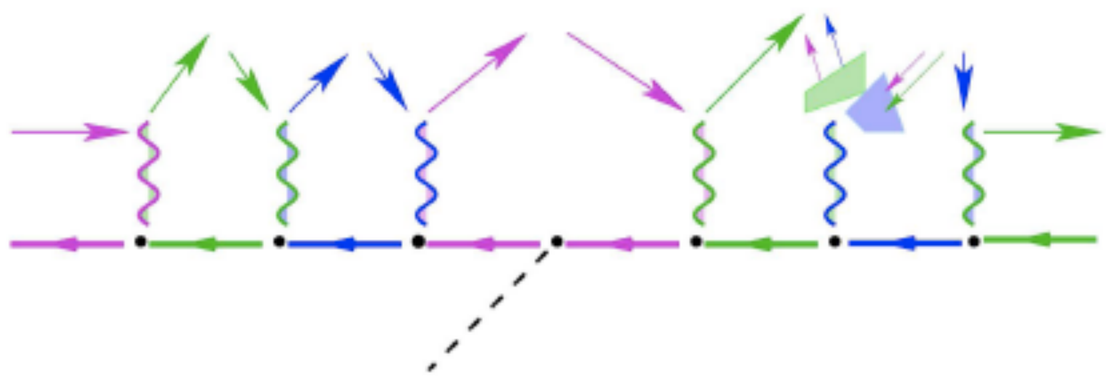


- A *green–anti-green* quark pair pops up from the vacuum, splitting the system into two globally blanched sub-systems.



"One" hadron per $\frac{\Delta\omega}{\omega}$; Hadron multiplicity $\propto \ln Q$.

Phenomenological realization of the Kogut–Susskind scenario



$\Longrightarrow$ a "String" of hadrons

The base of the Lund Model

The key features of the Lund hadronization model:

- Uniformity in *rapidity*: $dN_h = \text{const} \times \dfrac{d\omega_h}{\omega_h}$
- Limited $k_\perp$ of hadrons
- Quark combinatorics at work
  - *u*, *d* vs. *s*
  - *mesons* vs. *baryons*

The crucial step: Stress on the *rôle of colour* in multiple hadroproduction

QCD possesses $N_c^2 - 1$ gauge fields — vector gluons g.

At large distances, they are supposed to "glue" quarks together.

At small distances (space-time intervals) g is as legitimate a parton as q is.

The first indirect evidence in favour of *gluons* came from DIS where it was found that the electrically charged partons (quarks) carry, on aggregate, *less than 50%* of the proton's energy–momentum.

Now, we see a gluon emitted as a "real" particle.

What sort of final hadronic state will it produce?

B.Andersson, G.Gustafson & C.Peterson, Lund Univ., Sweden (1977)

Gluon $\simeq$ quark-antiquark pair:

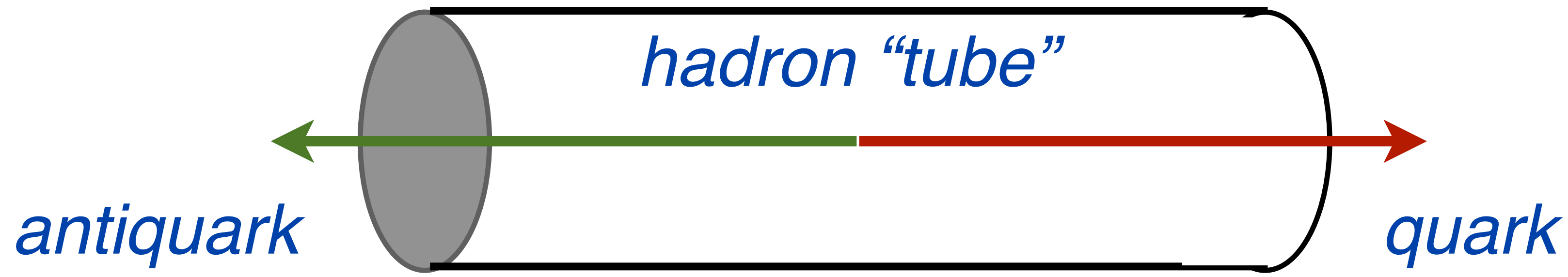
$3 \otimes \bar{3} = N_c^2 = 9 \quad \simeq \quad 8 = N_c^2 - 1.$

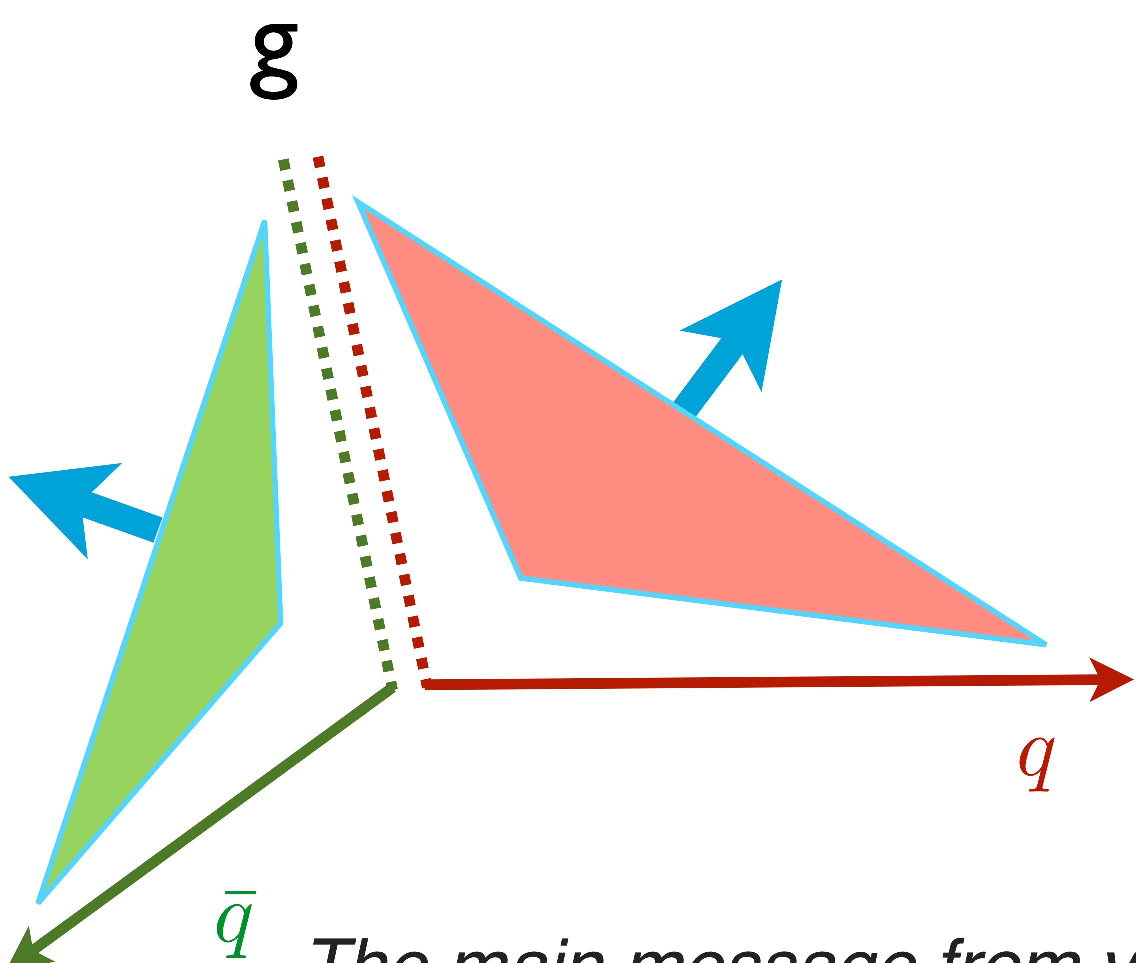
Relative mismatch : $\mathcal{O}(1/N_c^2) \ll 1$ (the large-N_c limit)

Lund model interpretation of a *gluon* —

Gluon – a "*kink*" on the "string" (colour tube)
that connects the quark with the antiquark

Lund model





Lund "String effect"

kinematical effect of a boosted hotdog

Lund's stress on ***topology*** of the dominant ***color flow*** found support from **p**QCD

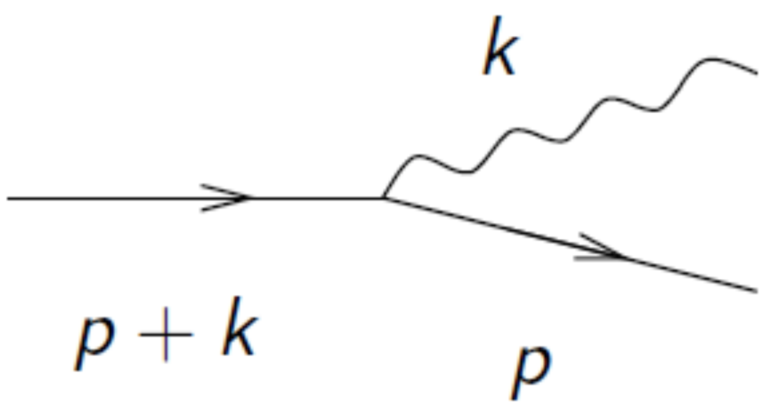
The main message from various manifestations of "QCD Radiophysics"

*- the yield of final state **hadrons***
is proportional to the Poynting vector
*of the underlying soft **gluon** field !*

To answer the question as how do offspring partons influence the hadronic yield, one has to realize first ***what is the condition*** for a gluon to behave as an ***independent colored object*** and thus as an ***additional source of new particles***.

It takes time to emit a gluon.

The formation time can be simply estimated as a ***lifetime*** of the virtual **(p + k)** quark state



Making use of the Heisenberg uncertainty principle, with account of the Lorentz contraction,

$$t_g^{\text{form}} \sim \frac{1}{M_{virt}} \cdot \frac{E}{M_{virt}} = \frac{E}{(p+k)^2} \approx \frac{E}{kE\Theta^2} \approx \frac{k}{k_\perp^2}$$

Comparing with the hadronization time, $t_g^{\text{hadr}} \approx kR^2$,

$$t_g^{\text{form}} \sim \frac{k}{k_\perp^2} \quad < \quad t_g^{hadr} \sim kR^2$$

the gluon's being is guaranteed *iff* its transverse momentum is large:

$$k_\perp > R^{-1} = \text{a few hundred MeV.}$$

This "***get-born-before-dying***" condition goes along with the ***coupling*** behaviour : $\alpha_s(k_\perp^2)$

- $R^{-1} \ll k_\perp \ll k \sim \sqrt{Q^2}$

- the domain of ***quasi-collinear*** hard parton splittings leading to the scaling violation effects in DIS structure functions and jet fragmentation (inclusive particle distributions)

- $R^{-1} \ll k_\perp \sim k \ll \sqrt{Q^2}$

- ***large angle soft*** gluon emission responsible for drag effects in interjet multiplicity flows

- $R^{-1} \ll k_\perp \ll k \ll \sqrt{Q^2}$

- ***double-logarithmic*** (soft & collinear) gluon bremsstrahlung off quarks and gluons causing jet multiplicity grow with energy and determining QCD parton Form Factors

All these are legitimate, PT-controllable, QCD sub-processes.

Parton pairs with small relative transverse momenta lie beyond PT control.

Let us look at the gluons radiated at the ***lower edge*** of the PT phase space : $k_\perp \sim R^{-1}$.

An appearance of a "**gluer**" is a signal of switching on of the ***real strong interaction*** :

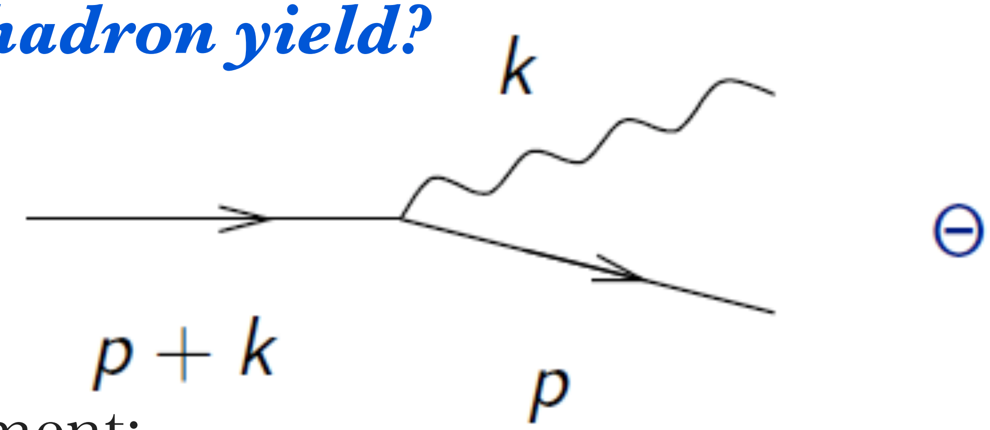
$$\left(t^{\mathrm{form}} \sim t^{\mathrm{hadr}}\right)_{\mathrm{gluer}}$$

Inclusive spectrum of **gluers** makes one think of the ***Feynman hadron plateau*** :

$$dN = \left[\int_{k_\perp \sim R^{-1}} \frac{dk_\perp^2}{k_\perp^2}\, 4C_F \frac{\alpha_s(k_\perp^2)}{4\pi}\right] \frac{dk}{k} \quad = \mathrm{const} \cdot \frac{dk}{k}$$

gluers and hadronization

How will an additional PT gluon contribute to the hadron yield?



Look at the time when the secondary gluon and its parent separate in the transverse plane at a critical confinement distance $t^{\text{separ}} \cdot c\Theta \simeq \Delta\rho_{\perp} \sim R.$

We expect strong interaction to enter the stage at this moment:

- vacuum break-up
- production of a hadron (or a few)
- colour blanching of separating objects a'la Kogut–Susskind

$$t^{\text{form}} \approx k/k_{\perp}^{2}$$

$$t^{\text{separ}} \approx R/\Theta = t^{\text{form}} * (k_{\perp}R)$$

$$t^{\text{hadr}} \approx kR^{2} = t^{\text{form}} * (k_{\perp}R)^{2}$$

At this very moment a gluer is formed, with $k_{\text{gluer}} \sim (R\Theta)^{-1}$

$$\left(t^{\text{form}} \sim t^{\text{separ}} \sim t^{\text{hadr}}\right)_{\text{gluer}} \approx R/\Theta = (t^{separ})_{qg}$$

which ensures separation of partons as ***globally blanched sub-jets***

two important messages

The first (softest) hadron that appears in course of ***q-g*** separation is ***quite energetic***:

$$\omega_h \sim k_{\rm gluer} \sim (R\Theta)^{-1} \gg R^{-1}.$$

This is the effect of the "Lorentz boost" of the ***qg*** system, ***provided*** it is the development of the ***color field*** that is responsible indeed for the production of hadrons !

The gluon sub-jet will develop plateau of hadrons with energies $(R\Theta)^{-1} < \omega_h < k.$

The length of the "additional plateau" of hadrons is $\eta = \ln k/(R\Theta)^{-1} = \ln k_\perp$ and so is the hadron multiplicity due to the gluon radiation !

an anti-Field–Feynman picture !

The PT gluon ***k*** and its parent quark ***p*** will have to ***hadronize independently*** since the distance between partons at time t_{hadr} is large :

$$\begin{aligned} t^{\rm separ} &\approx R/\Theta = t^{\rm form} * (k_\perp R)\,, \\ t^{\rm hadr} &\approx kR^2 = t^{\rm form} * (k_\perp R)^2\,. \end{aligned}$$

Hadronization must be ***local*** in the configuration space.

Qualitative space-time analysis of parton cascades accompanied by the example of the ***one-particle inclusive spectrum*** whose shape at small ***x*** was found to be ***insensitive, at the PT level, to large-distance phenomena*** gave rise to the idea of ***mathematical similarity*** between calculable parton and observable hadron distributions :

"Asymptotic Freedom and Local Parton–Hadron Duality" **(1984)**

The two ***fragmentation models*** that have survived the pressure of LEP scrutiny

— the **Lund string** (Andersson, Gustafson)
— the **HERWIG cluster** (Marchesini,Webber) do respect the locality and the LPHD :

Lund by construction (universal fragmentation of the color tube)
HERWIG by virtue of finiteness of $\mathbf{M^2}$ of neighboring partons.

For quite some time in the 80s, LUND and HERWIG seemed "*orthogonal* ".

LUND was ignoring the ***PT parton multiplication*** until it got hard-pressured by LEP.

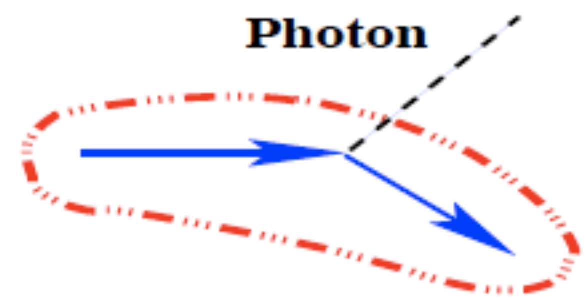
Good start : we learned how much of what we observe in jet physics can be explained merely by an accurate account of the momentum conservation and Lorentz kinematics.

On the other hand, the HERWIG ideologues from the start put an emphasis on the PT cascades, having chosen the PT–NP separation parameter Q_0 as low as possible.
At the same time, HERWIG had to learn, after the LUND, how to embed important effects of ***color topology*** upon large-angle particle production : the ***inter-jet phenomena***

Convergence : "The string effect and QCD coherence", Phys.Lett. 165B (1985) 147

QCD Radiophysics

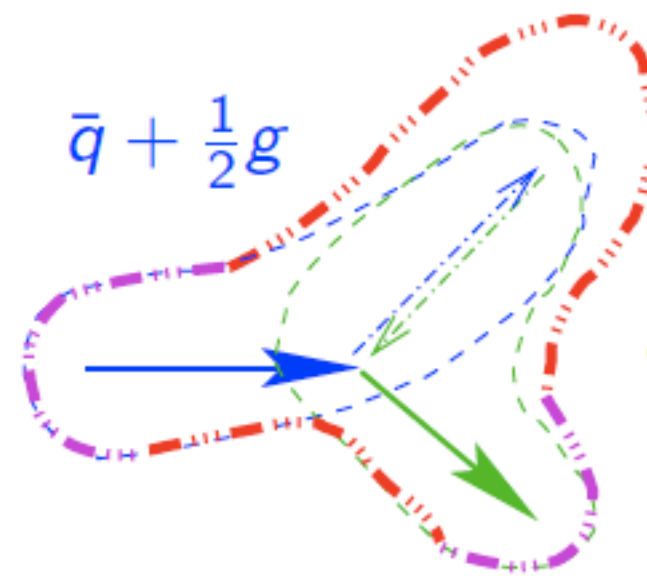
Look at hadrons produced in a $q\bar{q}$+photon e^+e^- annihilation event



Now substitute a gluon for the photon in the same kinematics

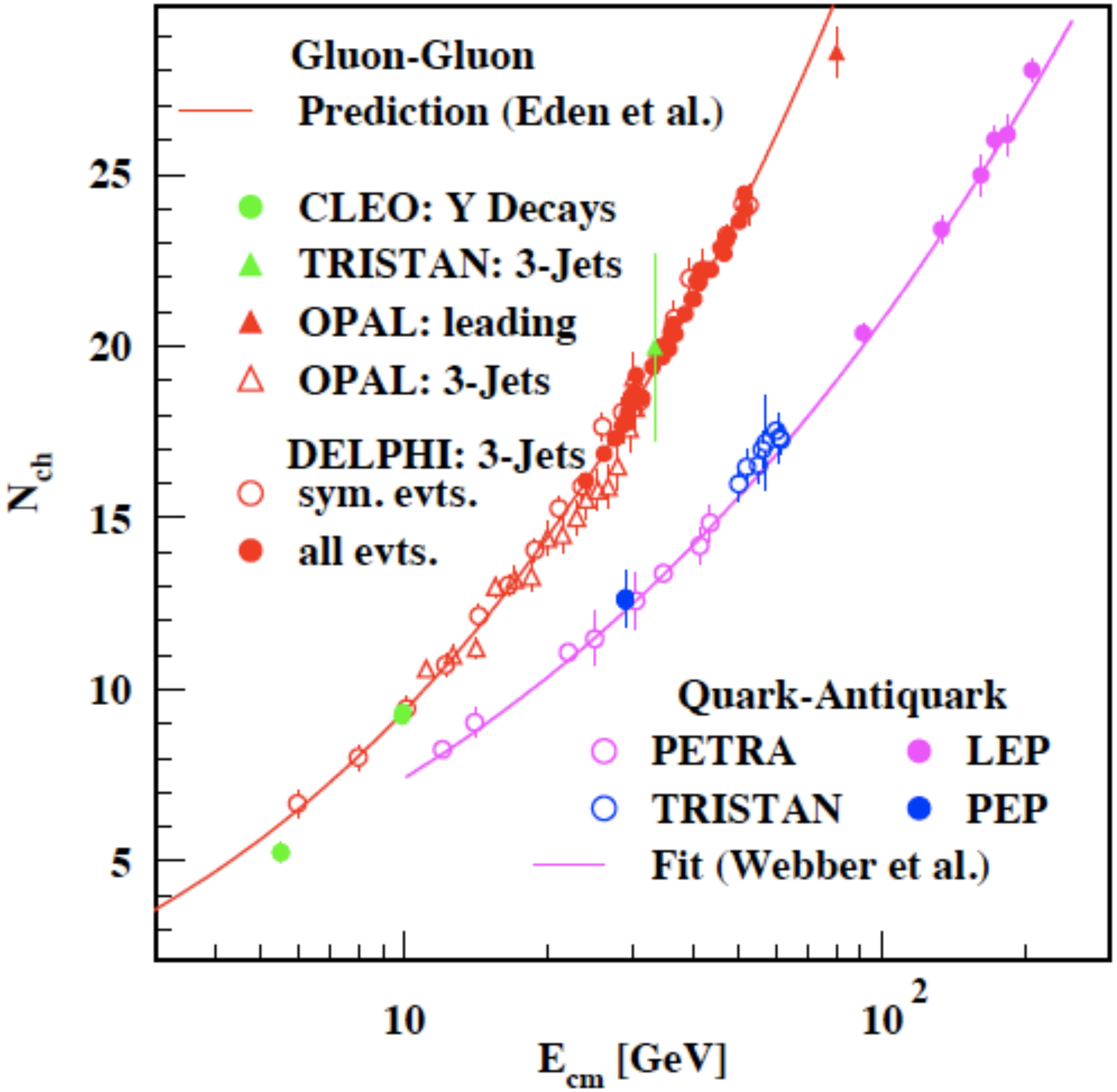
The gluon carries "double" colour charge;
quark pair is *repainted* into octet colour state

Lund: hadrons = the sum of two independent (properly boosted) colorless substrings



The first immediate consequence :

Double Multiplicity of hadrons in fragmentation of the *gluon*



Look at experimental findings

Lessons :

- N increases *faster* than $\ln E$

  ($\Longrightarrow$ Feynman was wrong)

- $N_g / N_q \; < \; 2$

however

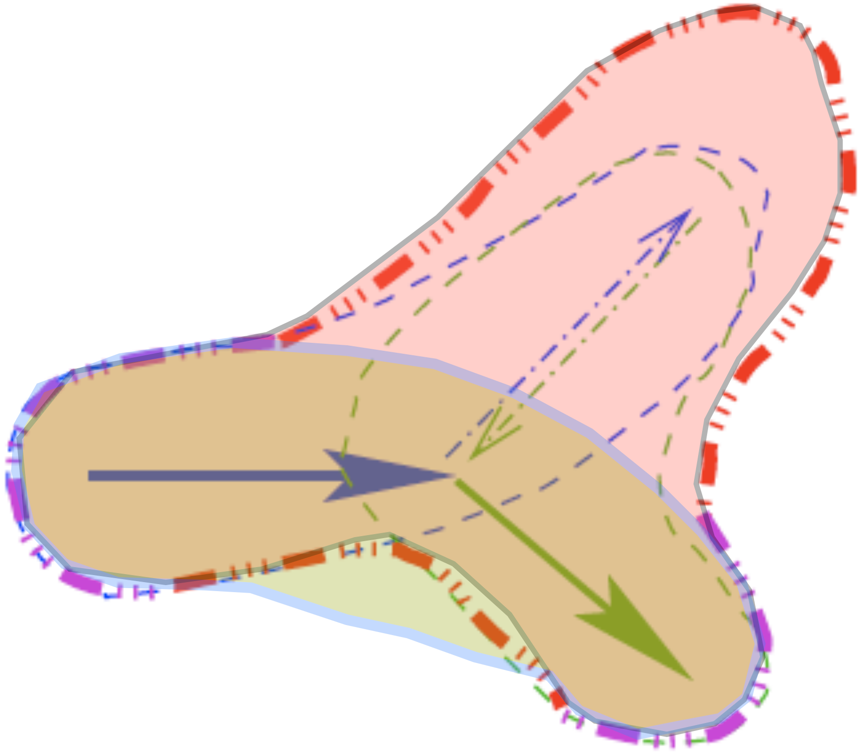
look at the ***energy derivative*** :

- $\frac{dN_g}{dN_q} = \frac{N_c}{C_F} = \frac{2N_c^2}{N_c^2-1} \; = \frac{9}{4} \simeq 2$

It is bremsstrahlung gluons who add to the hadron yield

Parton cascades respect QCD "color counting"

Now return to a more subtle - and powerful - consequence of the Lund picture



Lund : final hadrons are given by the sum of two independent substrings made of

$$q + \tfrac{1}{2}g \quad \text{and} \quad \bar{q} + \tfrac{1}{2}g$$

Let's look into the *inter-quark valley* and compare the hadron yield with that in the $q\bar{q}\gamma$ event

The overlay results in a magnificent "*String effect*"

depletion of particle production in the $q\bar{q}$ valley

Destructive interference from the QCD point of view

$$\text{QCD prediction}: \quad \frac{dN_{q\bar{q}}^{(q\bar{q}\gamma)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{2(N_c^2-1)}{N_c^2-2} = \frac{16}{7} \qquad (\text{experiment: } 2.3 \pm 0.2)$$

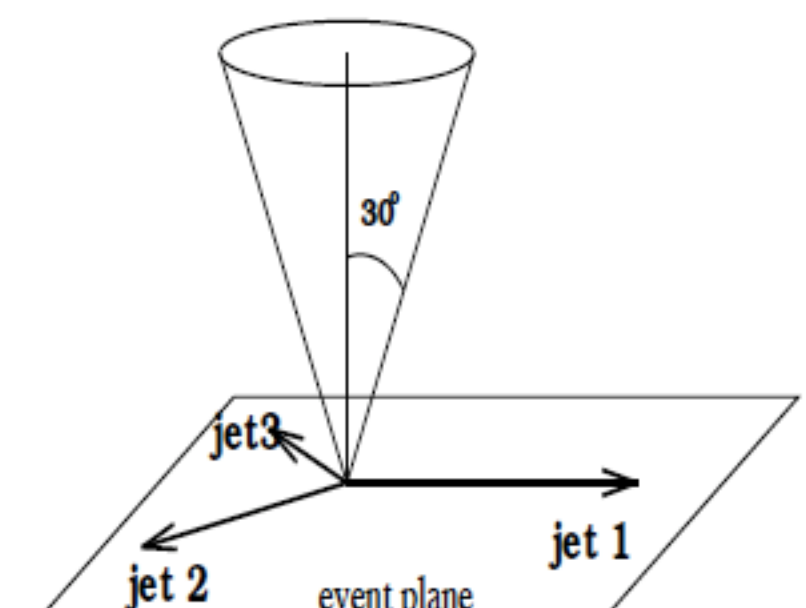
Multitude of ***ratios of hadron flows*** between jets in various multi-jet processes gives example of a non-trivial CIS (collinear-and-infrared-safe) QCD observable

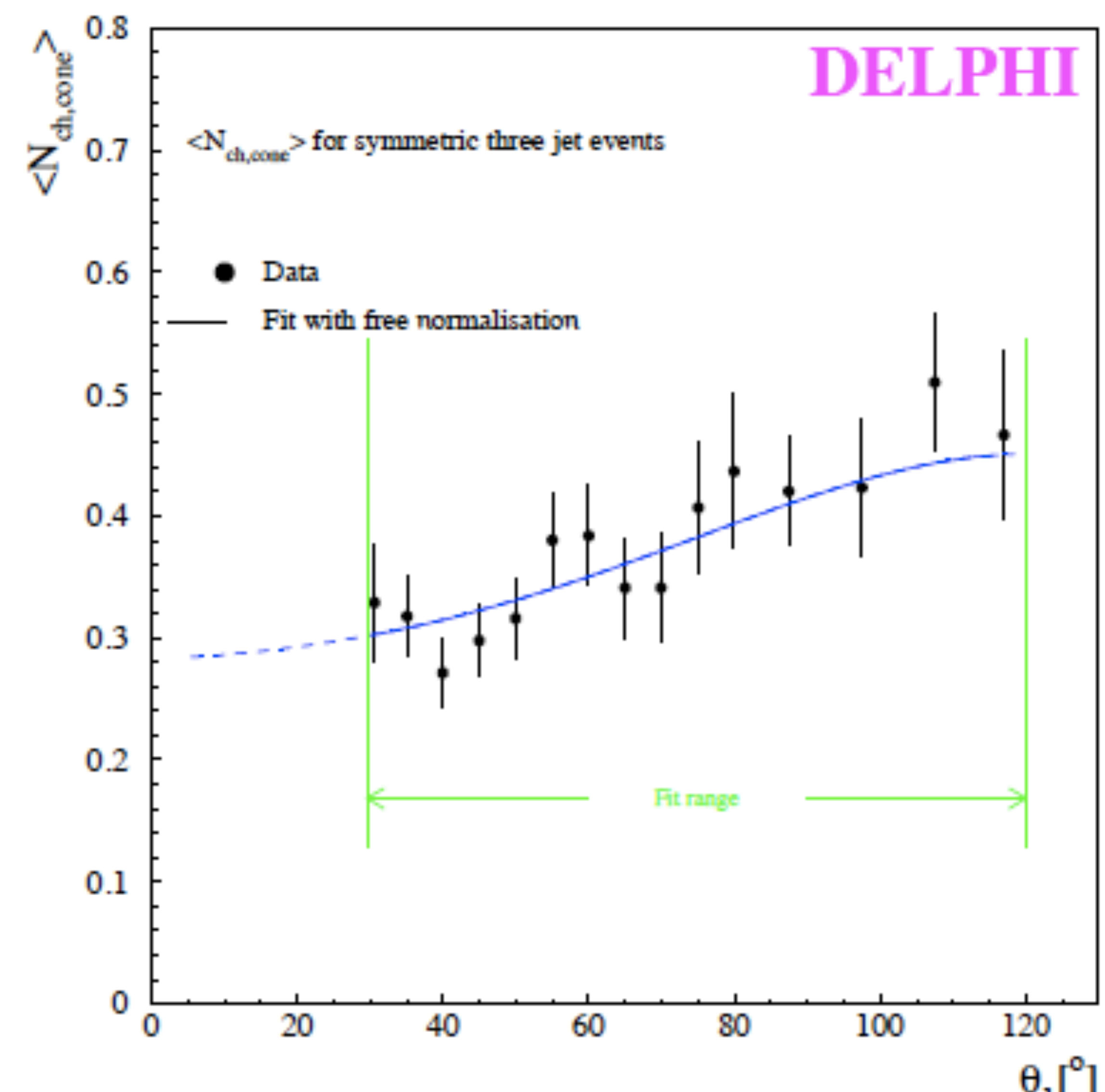
The classical *string effect* – the ratio of the multiplicity flow between a quark (antiquark) and a gluon to that in the quark valley in symmetric ("Mercedes") three-jet e+e– annihilation events :

$$\frac{dN_{qg}^{(q\bar{q}g)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{5N_c^2-1}{2N_c^2-4} \quad = \frac{22}{7} \simeq \pi$$

Emitting an energetic gluon off the initial quark pair depletes accompanying radiation in the *backward* direction: color is *dragged out* of the quark valley. This destructive interference effect is *so strong* that the resulting multiplicity flow between quarks falls below that in the *least favorable* direction - *transversal* to the 3-jet event plane :

$$\frac{dN_{\perp}^{(q\bar{q}\gamma)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{N_C+2C_F}{2(4C_F-N_c)} = \frac{17}{14}$$





The *color field* that an ensemble of hard primary ***partons*** (parton antenna) develops, determines, on the "one-to-one" basis, the structure of final flows of ***hadrons***.

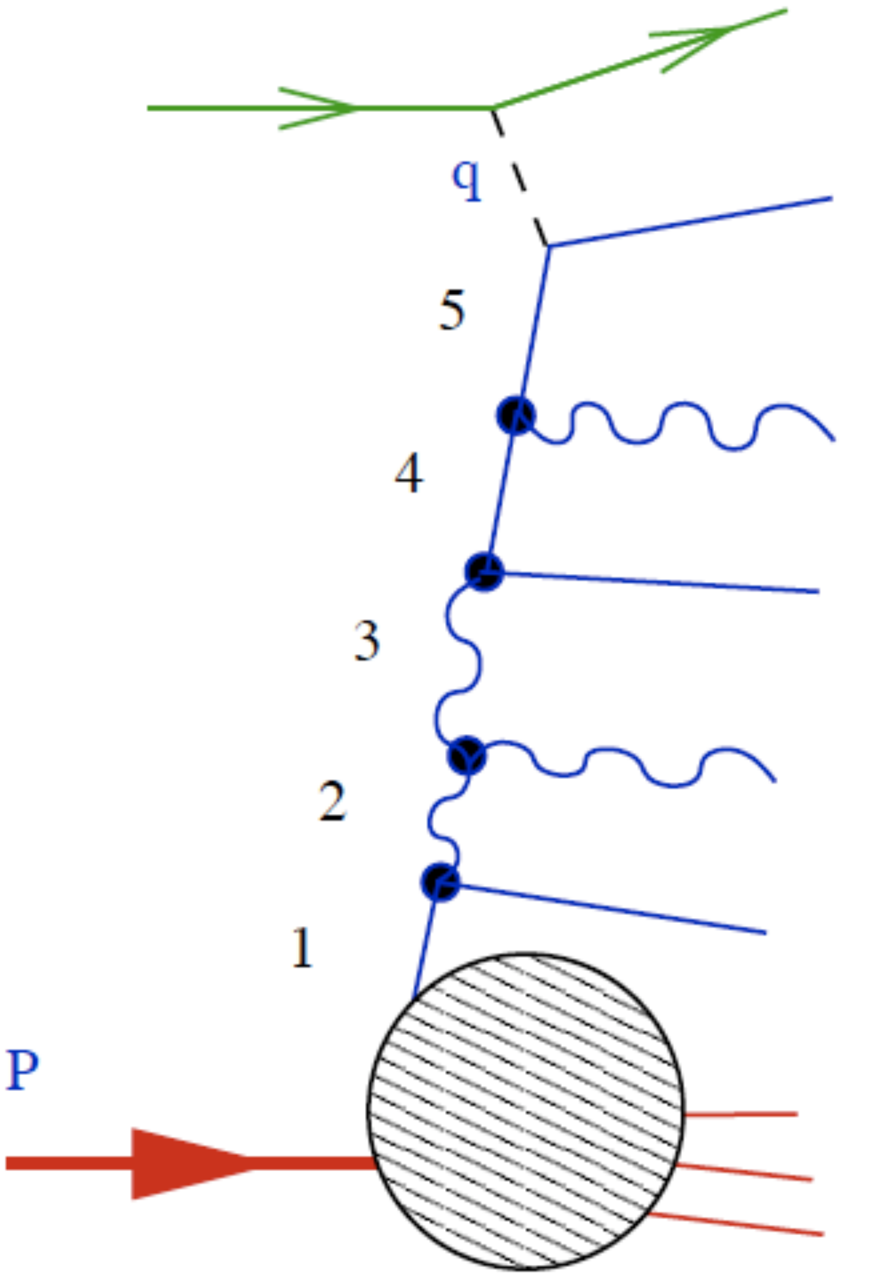
The Poynting vector of the ***color field*** gets "*translated*" into the ***hadron*** Poynting vector :

When viewed ***globally***, confinement is about "***renaming***" a flying-away **quark** into a flying-away **pion** rather than about forces "***pulling***" quarks together.

QCD coherence is crucial
for multiplication of partons
inside jets as well.

**Probabilistic parton evolution picture
beyond the LLA**

***quark–gluon cascades** (space-like)*



So long as probability of one extra parton emission is large, one has to consider and treat *arbitrary number* of parton splittings

$$\frac{P}{\mu^2} \gg t_1 \gg t_2 \gg t_3 \gg t_4 \gg t_5 \gg \frac{P}{Q^2}$$

$$\mu^2 \ll k_{1\perp}^2 \ll k_{2\perp}^2 \ll k_{3\perp}^2 \ll k_{4\perp}^2 \ll k_{5\perp}^2 \ll Q^2$$

"Hamiltonian" for parton cascades

$$\Phi_q^q(z) = C_F \cdot \frac{1+z^2}{1-z},$$

$$\Phi_q^g(z) = C_F \cdot \frac{1+(1-z)^2}{z},$$

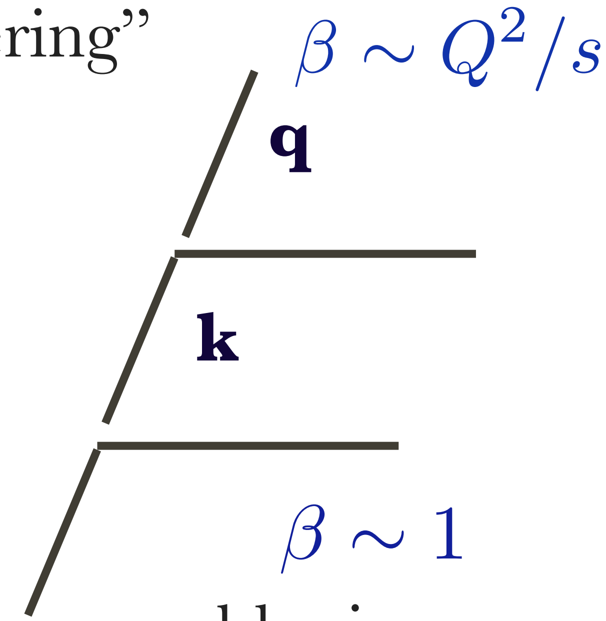
$$\Phi_g^q(z) = T_R \cdot \left[z^2+(1-z)^2\right],$$

$$\Phi_g^g(z) = N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

Why does one order successive parton splittings in transverse momenta rather than in the fluctuation time ?

Analyzing Feynman denominators, we came to "fluctuation time ordering"

$$t_{[q]} \simeq \frac{\beta_q}{q_\perp^2} \ll \frac{\beta_k}{k_\perp^2} \simeq t_{[k]}$$



If we used this in earnest, the evolution equations would have become ***non-local*** since the dependence of ***pdf***s on **Q** and the dependence on **x** would mix ...
In fact, no-one does so. ***Why ?***

In the LLA (1-loop level), the beta-factors can be dropped altogether.
However, nowadays the DIS phenomenology employs the ***three-loop*** parton dynamics **!**
Already ***beyond the 1st loop*** (LLA), it starts to matter how to order successive parton splittings that is, which variable to choose for "***parton evolution time***".
This choice affects ***higher-loop*** "anomalous dimensions" - our "*Hamiltonian*".

The "***clever choices***" had been established quite some time ago:

Transverse momentum ordering (Gribov & Lipatov)
space-like parton evolution; DIS structure functions

$$d\xi = d\ln\frac{k_\perp^2}{1}$$

Angular ordering (Fadin & Mueller)
time-like parton multiplication; jet fragmentation functions

$$d\xi = d\ln\frac{k_\perp^2}{z^2}$$

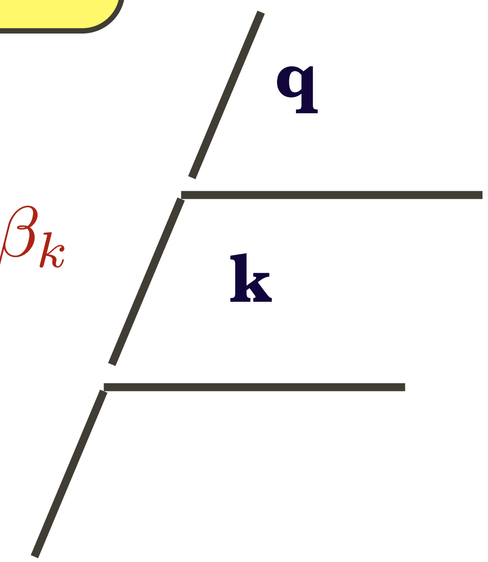
Each is a consequence of taking into full consideration soft gluon coherence to prevent explosively large terms $(\alpha_s \ln^2 x)^n$ from appearing in higher loop anomalous dimensions.

Space-like parton evolution: DIS at small Bjorken x

$$t_{[q]} \simeq \frac{\beta_q}{q_\perp^2} \ll \frac{\beta_k}{k_\perp^2} \simeq t_{[k]}$$

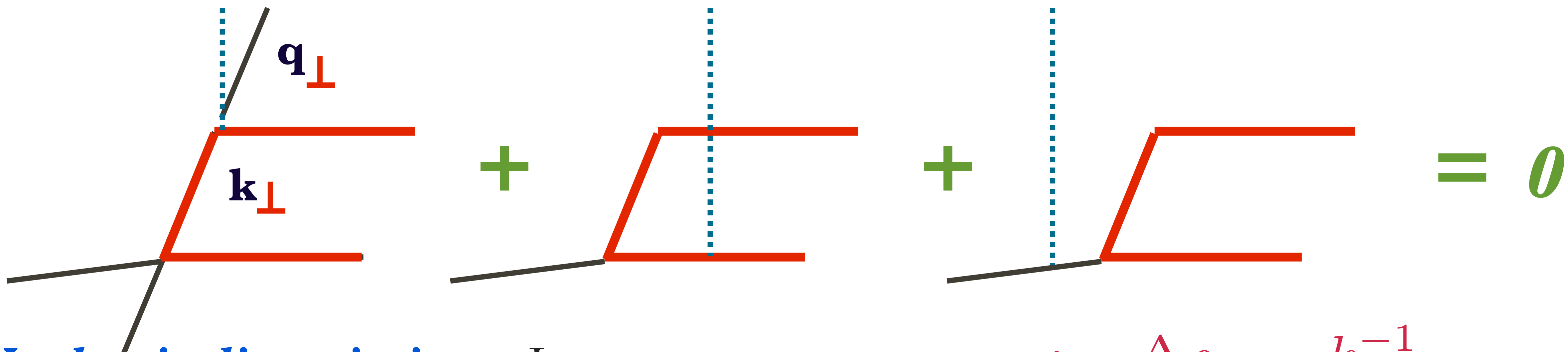
$k_\perp$ integration phase space swells when $\beta_q / \beta_k \longrightarrow 0$

$\beta_q \ll \beta_k$



BFKL ***(high energy hard scattering)*** $\alpha_s \int \frac{d\beta}{\beta} \Longrightarrow \alpha_s \ln s$

NO $\alpha_s \ln^2 s$ terms… *How has this happened?* Quantum-mechanical coherence at work !



Inelastic dissociation. Into a ***compact*** state : $\Delta\rho_\perp \sim k_\perp^{-1}$

In a ***long-range*** potential : $\lambda_\perp \sim q_\perp^{-1} \gg \Delta\rho_\perp$

In QCD/QED vanishing of forward inelastic processes follows from gauge invariance.

In a more general context, it is due to orthogonality of the initial and final state wave functions, *provided* the initial and final systems *interact identically* with “the probe”.

evolution time

We see that the "kinematical" *fluctuation time ordering* seems to be of little relevance as it misses essential physics.

$$t_{[q]} \simeq \frac{\beta_q}{q_\perp^2} \ll \frac{\beta_k}{k_\perp^2} \simeq t_{[k]}$$

Thus, in order to take into consideration destructive interference effects *we have to replace*

ordered lifetimes

$$\frac{P}{\mu^2} \gg t_1 \gg t_2 \gg t_3 \gg t_4 \gg t_5 \gg \frac{P}{Q^2}$$

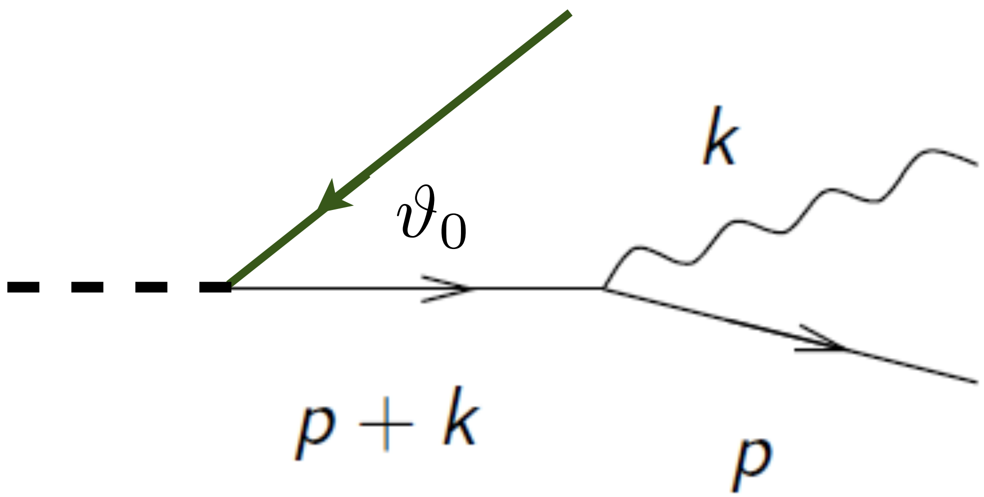
by ordered transverse momenta

$$\mu^2 \ll k_{1\perp}^2 \ll k_{2\perp}^2 \ll k_{3\perp}^2 \ll k_{4\perp}^2 \ll k_{5\perp}^2 \ll Q^2$$

Formally, such a step is unnecessary (though convenient) to take in the Leading Log Approximation where we keep track, exclusively, of the leading ***transverse momentum*** *logarithms.*

In higher orders, in becomes ***mandatory*** *as it avoids appearance of superficial double logarithms in the anomalous dimension of DIS at* $x \ll 1$.

Production of relatively soft gluons in ***time-like*** cascades (jets)



Θ

Formation time of the gluon :

$$t_{\text{form}} \simeq \frac{k}{k_\perp^2} = \lambda_\perp \cdot \frac{1}{\Theta}$$

Let us look at the ***distance*** between the radiating ***color charges.***

During this time they separate in the transverse plane by $\Delta\rho_\perp \simeq tc \cdot \vartheta_0$

Now compare this separation with the ***wavelength*** of the radiation : $\frac{\Delta\rho_\perp}{\lambda_\perp} = \frac{\vartheta_0}{\Theta}$

If this ratio is ***larger than 1***, $\Theta < \vartheta_0$, the gluon will be radiated ***independently*** by the two partons of the previous generation.

Otherwise, gluon radiation intensity will be determined by the ***overall color charge*** of the parton pair. ***Soft gluon coherence!***

Probabilistic parton cascades → ***Angular Ordering***

intra-jet coherence

Destructive interference suppresses multiple production of *very small momentum* gluons.
It is particles with ***intermediate energies*** that happen to multiply most efficiently !

The energy spectrum of relatively soft secondary partons in jets develops a "***hump*** ".
The position, the width and the height of this hump evolve with $\mathbf{Q^2}$ in a predictable way.

This prediction was derived in 1984 in the so-called MLLA (Modified Leading Log Approximation).
It took into account essential ingredients of parton multiplication in the next-to-leading order :

parton ***splitting functions*** responsible for the energy balance in parton splitting

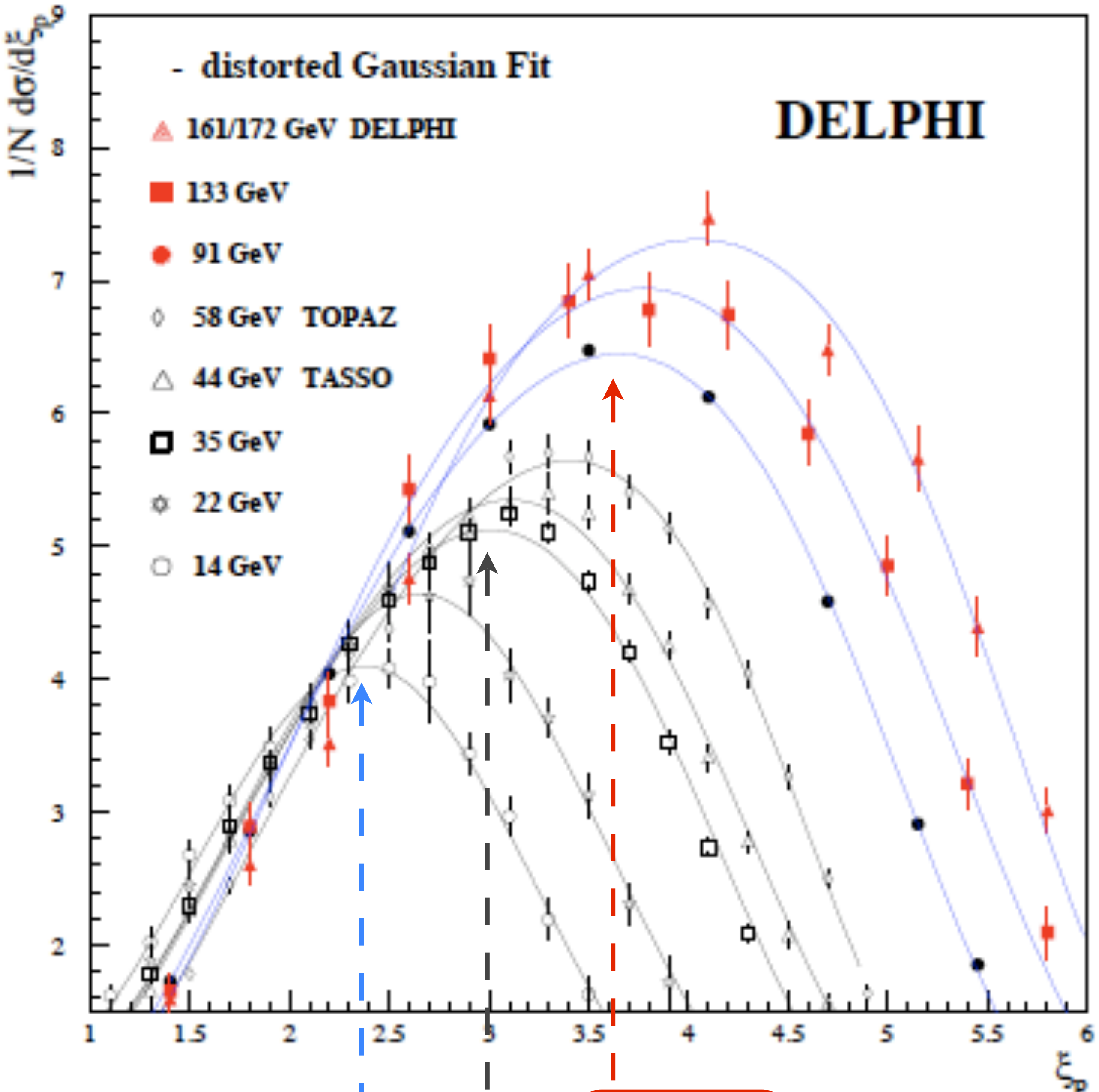
the ***running coupling*** depending on the *relative transverse momentum* of the offspring

the ***exact angular ordering.*** The latter - a consequence of ***soft gluon coherence***.

Moreover, in turned out that "**soft jet fragmentation**" can be predicted, in certain sense, ***from the first principles*** unlike **DIS** parton distributions where one always needs a **NP** input.

The shape of the inclusive spectrum of *all charged* ***hadrons*** (dominated by pions)
exhibits the same features as the MLLA **parton** spectrum

First scrutinized at LEP, the similarity of *parton* and *hadron* energy spectra has been verified at SLC and KEK $\mathbf{e^+e^-}$ machines, as well as at HERA and Tevatron (where jets originate not from quarks dug up from the vacuum by a virtual photon/Z^0 but from partons kicked out from initial hadron(s)).



The comparison of the spectra of all charged hadrons at various annihilation energies Q with the so-called "*distorted Gaussian*" fit which employs the first four moments (the mean, width, skewness and kurtosis) of the MLLA distribution around its maximum.

Shall we say : a (routine, interesting, wonderful) check of yet another QCD prediction?

Such a close similarity offers a ***deep puzzle***, even a worry, rather than a successful test

The observation of the parton-hadron similarity was initially met with a serious scepticism: it looked more natural to blame the ***finite hadron mass effects*** for falloff of the spectrum at large ξ (small momenta) rather than seriously believe in applicability of the PT consideration down to such ***disturbingly small scales***.

As we have already discussed above looking unto gluon formation physics, it is ***not the energy*** of the jet but ***the maximal parton transverse momentum*** inside it, $k_{\perp\max} \simeq E_{\rm jet} \sin\Theta_{\rm cone}$ that determines the hardness scale and thus the yield and the distribution of the accompanying radiation.

This means that by choosing a small opening angle one can study relatively small hardness scales **but in a cleaner environment** : due to the ***Lorentz boost effect***, eventually all particles that form a short small-Q^2 QCD "**hump**" are now ***relativistic*** and are concentrated ***at the tip of the jet !***

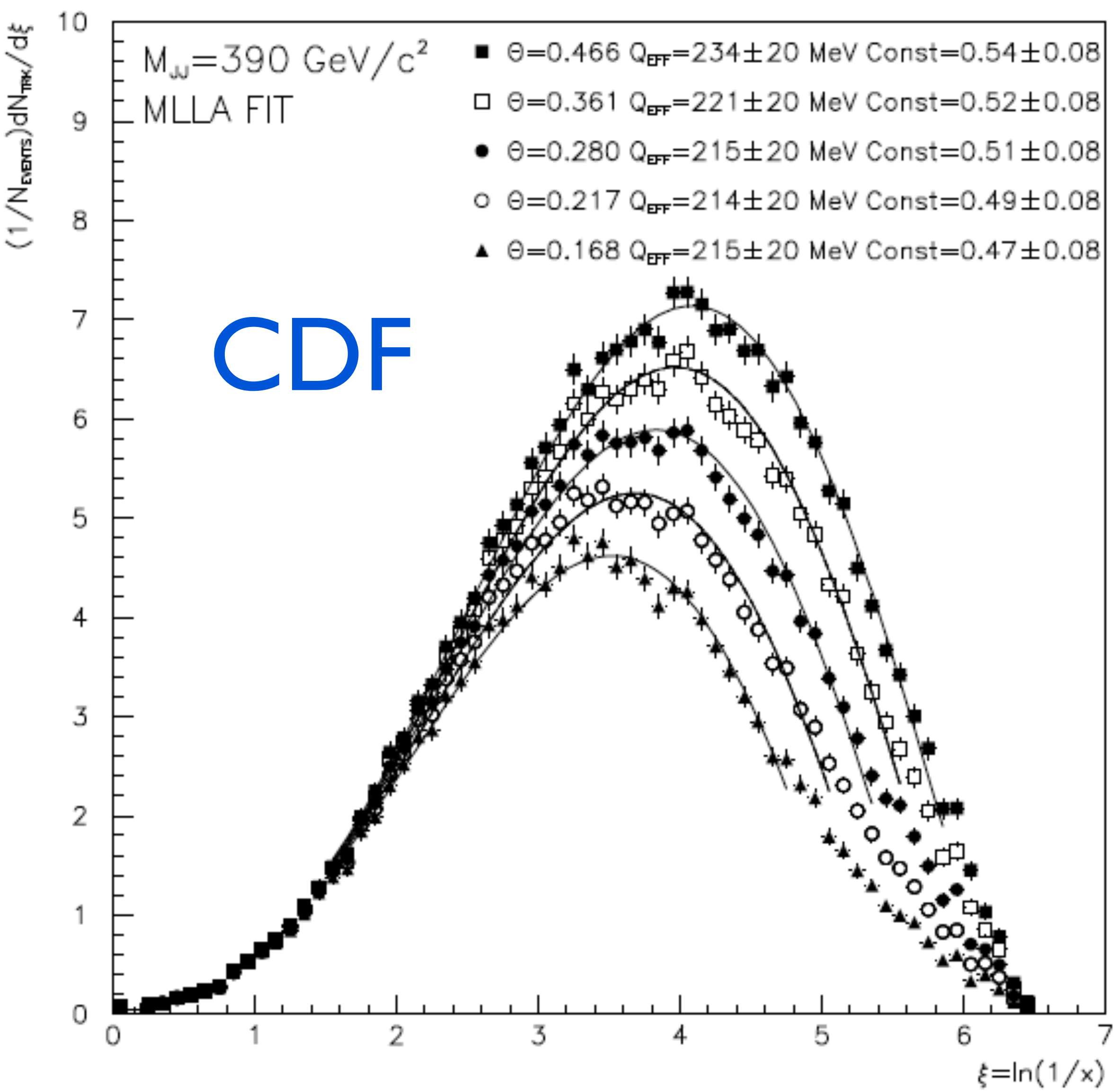
Selecting hadrons inside a cone **0.14** around a quark jet with $E_{jet}=100$ **GeV** one would see that very *dubious* $Q=14$ **GeV** curve but now with the maximum ***boosted*** from 0.45 **GeV** to 6 **GeV !**

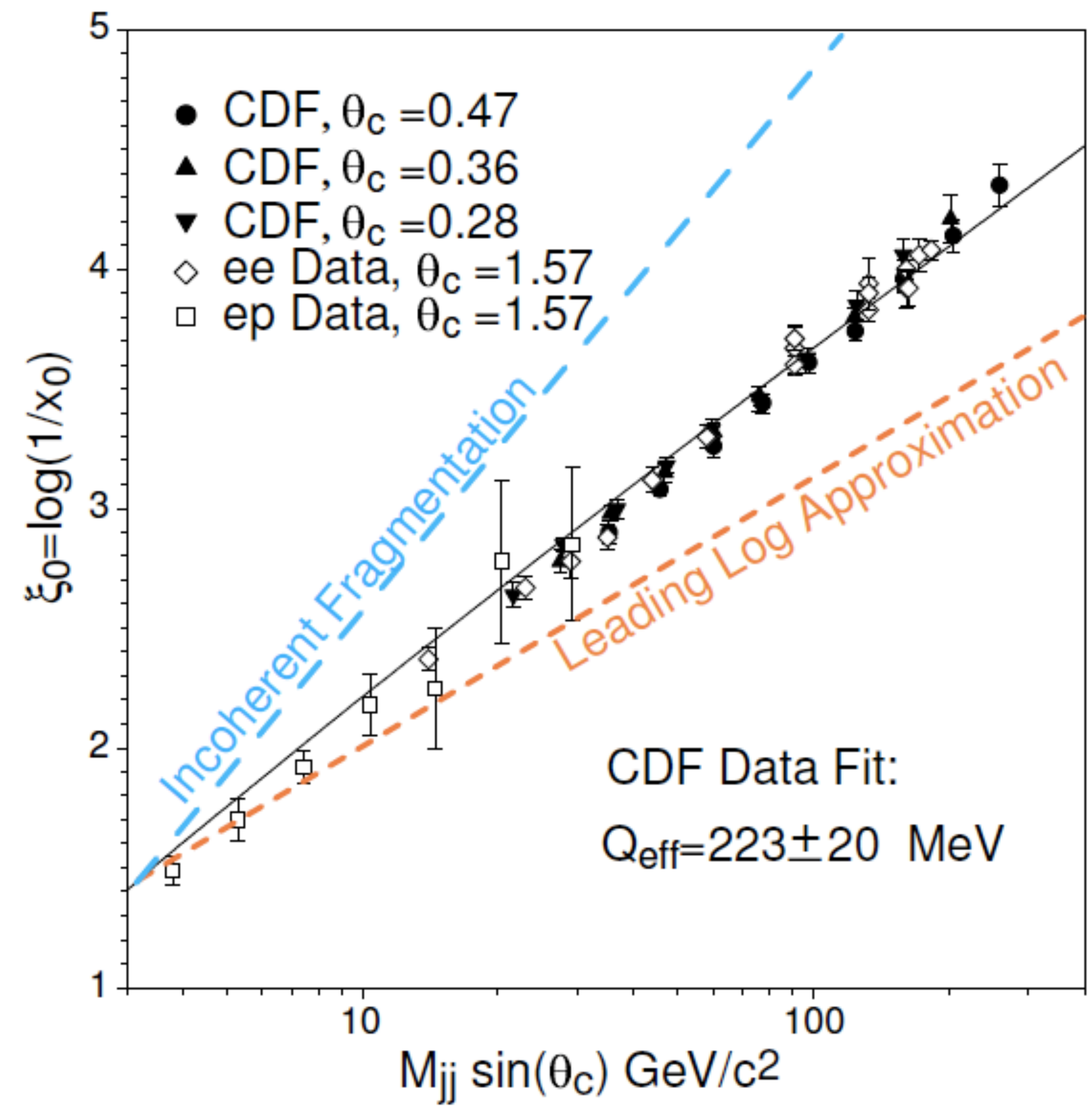
jets with restricted "opening angle"

Position of the Hump as a function of the ***hardness*** of the jet $Q = M_{jj} \sin\Theta_c$

is a ***parameter-free*** *pQCD prediction*

The plot combines ***e^+e^-***, ***DIS*** and ***hh*** data **!**





punchline

We have seen that the ratios of particle flows between jets (intERjet radiophysics), as well as the shape of the inclusive spectra of secondary particles (intRAjet cascades) turn out to be formally calculable (CIS) quantities.

Moreover, the perturbative QCD predictions actually describe flows of **hadrons** !

The strange thing is, these phenomena reveal themselves in present-day experiments via hadrons (pions) with extremely small momenta $k_\perp$, where we are expecting to hit the non-perturbative domain — large coupling $\alpha_s(k_\perp)$ — and potential ***failure*** of the *quark–gluon language* as such.

The fact that the underlying perturbative dynamics of color is being impressed upon "miserable" pions with 100–300 MeV momenta, could not be a priori expected.

At the same time, it repeatedly sends us a powerful message: ***confinement*** – *transformation of quarks and gluons into hadrons* – has a ***non-violent*** nature: there is no visible reshuffling of energy–momentum at the hadronization stage.

Known under the name of the Local Parton-Hadron Duality hypothesis (LPHD), explaining this phenomenon remains a challenge for the future quantitative theory of color confinement.

"SOFT CONFINEMENT" . . .

EXTRAS

DIS evolution

Recall the ***evolution quation*** for the space-like parton cascades (DGLAP)

$$\frac{\partial}{\partial \ln k_\perp} D(x,k_\perp) = \frac{\alpha_s(k_\perp)}{\pi} \int_x^1 \frac{dz}{z}\, P(z)\, D\left(\frac{x}{z}, k_\perp\right)$$

The ***Mellin transform*** is like trading the coordinate (**ln x**) for momentum in Schroedinger equation

$$D_\omega = \int_0^1 dx\, x^\omega \cdot D(x)\,, \qquad D(x) = x^{-1}\int_{(\Gamma)} \frac{d\omega}{2\pi \mathrm{i}}\, x^{-\omega}\cdot D_\omega$$

$$P_\omega = \int_0^1 dz\, z^\omega \cdot P(z)\,, \qquad P(z) = z^{-1}\int_{(\Gamma)} \frac{d\omega}{2\pi \mathrm{i}}\, z^{-\omega}\cdot P_\omega\,,$$

$$\hat{\mathrm{d}}\, D_\omega(k_\perp) = \frac{\alpha_s(k_\perp)}{\pi}\cdot P_\omega\, D_\omega(k_\perp)\,; \qquad \hat{\mathrm{d}} \equiv \frac{\partial}{\partial \ln k_\perp}$$

The "wavefunction" **D** is a multi-component object. It embodies the distributions of valence quarks, as well as gluons and secondary sea quarks which evolve and mix according the **2x2** matrix "Hamiltonian".

At small **x** the picture simplifies : the ***valence*** distribution is negligible, $O(\mathbf{x})$, while the ***gluon*** and ***sea quark*** components form a system of two coupled oscillators.

$$P_\omega = \frac{2N_c}{\omega} - a + \mathcal{O}(\omega)\,, \qquad a = \frac{11N_c}{6} + \frac{n_f}{3N_c^2}$$

What matters is a **singular** energy eigenvalue (*one of the two branches of the dispersion rule*).
Sea quarks are driven by the gluon distribution while the latter is dominated by gluon cascades.

$$D_\omega(k_\perp) = D_\omega(Q_0)\cdot \exp\left\{\int_{Q_0}^{k_\perp} \frac{dk}{k}\, \gamma_\omega(\alpha_s(k))\right\} \qquad \gamma_\omega(\alpha_s) = \frac{\alpha_s}{\pi} P_\omega$$

anomalous dimension

The **x**-dependence cannot be fully restored without the knowledge of the ***low-scale physics*** ...

if we choose to simply ignore it ,

$$xD(x,Q;Q_0) \simeq \int \frac{d\omega}{2\pi i} x^{-\omega} \exp\left(\frac{c}{\omega}\int_{Q_0}^{Q}\frac{dk}{k}\alpha_s(k)\right) \propto \exp\left(c'\sqrt{\ln x^{-1}\, \ln\frac{\ln Q/\Lambda_{QCD}}{\ln Q_0/\Lambda_{QCD}}}\right)$$

Angular ordering modifies the evolution equation :

$$\frac{\partial}{\partial \ln k_\perp} D(x, k_\perp) = \frac{\alpha_s(k_\perp)}{\pi} \int_x^1 \frac{dz}{z} P(z)\, D\left(\frac{x}{z}, z \cdot k_\perp\right) \qquad \frac{k'_\perp}{z} < \frac{k_\perp}{1} \quad \Leftrightarrow \quad \Theta < \vartheta_0$$

This equation is "non-local" but can be elegantly cracked using the Taylor expansion trick :

$$D(z \cdot k_\perp) = \exp\left\{ \ln z \frac{\partial}{\partial \ln k_\perp} \right\} D(k_\perp) = z^{\frac{\partial}{\partial \ln k_\perp}} \cdot D(k_\perp) \quad \Rightarrow \quad \hat{\mathrm{d}} \cdot D_\omega = \frac{\alpha_s}{\pi} P_{\omega + \hat{\mathrm{d}}} \cdot D_\omega$$

This leads to the ***differential equation*** approximately (DLA)

$$\left(P^{-1}_{\omega+\hat{\mathrm{d}}}\, \hat{\mathrm{d}} - \frac{\alpha_s}{\pi} - \left[P^{-1}_{\omega+\hat{\mathrm{d}}}, \frac{\alpha_s}{\pi} \right] P_{\omega+\hat{\mathrm{d}}} \right) \cdot D = 0 \quad \Rightarrow \quad (\omega + \gamma_\omega)\gamma_\omega - \frac{2N_c \alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s^2}{\omega} \right) = 0\,.$$

The anomalous dimension follows suit

$$\gamma_\omega = \frac{\omega}{2} \left(-1 + \sqrt{1 + \frac{8 N_c \alpha_s}{\pi\, \omega^2}} \right) = \frac{\alpha_s}{\pi} \frac{2N_c}{\omega} \left[1 + \sum_{k=1}^{\infty} c_k \left(\frac{\alpha_s}{\omega^2} \right)^k \right]$$

DL series in $\alpha_s \ln^2 x$

And now - **lo and behold !** - the region of "*smallish*" momentum scales where $\alpha_s(k'_\perp) \gg \omega^2$ affects the ***overall normalization*** of the parton yield but ***not the shape*** of the spectrum :

$$\gamma + \omega/2 \propto \sqrt{\alpha_s(k)} \neq f(\omega)$$

The ***shape*** of the energy spectrum of soft partons from QCD cascades turns out to be confinement-insensitive !